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MASS CONSERVING SCHEMES FOR SATURATED GROUNDWATER FLOW

E.W. Jenkins*, C. Paribello* and N.E. Wilson*

*Clemson University
Department of Mathematical Sciences
Martin Hall, Clemson, SC 29634 USA
e-mail: lea@clemson.edu, cparibe@clemson.edu, newilso@clemson.edu

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Summary. Local mass conservation is of great importance for accurate simulation of fluid flows, especially if the resulting velocity field is to be coupled to a transport equation. Previous work on simulation of saturated groundwater flows have used both mixed finite element methods and discontinuous Galerkin methods to achieve local mass conservation. However, legacy code using a continuous Galerkin finite element scheme must be rewritten to incorporate either of these methods. Recent research on so called *grad-div* stabilization schemes has shown they are capable of significantly improving mass conservation in Taylor-Hood approximations of the Stokes and Navier-Stokes equations. *Grad-div* stabilization schemes introduce one additional term into the governing equation. Another approach currently receiving a good deal of attention for approximating conservative solutions to the Navier-Stokes equations is the use of Scott-Vogelius approximating elements. These elements (i) require a barycenter refinement of a regular mesh, and (ii) generate an approximation which is point-wise mass conservative.

In this paper, we compare standard discretization schemes for the saturated flow equation with a *grad-div* stabilized scheme and a Scott-Vogelius scheme. We provide a comparison of the different schemes, including the degrees of freedom associated with each method, local mass conservation errors for each method, and convergence results.

1 INTRODUCTION

Global and local mass conservation for velocity fields associated with saturated porous media flow have long been recognized as integral components of any numerical scheme attempting to simulate these flows.[13,14,17,18] Several discretization approaches have successfully achieved this, including cell-centered finite difference methods,[14] mixed finite element methods,[6,14,19] and discontinuous Galerkin methods.[11,12,17,18] A-posteriori methods have also been used to project the computed velocity onto a divergence-free space.[4] While these methods are recognized as standards for local mass conservation, many legacy codes continue to use continuous Galerkin methods to simulate porous media flows.

In this paper, we are interested in solving the mixed form of the saturated flow equation

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\mathbf{u} = -\frac{k}{\mu}\nabla p + \mathbf{f}, \tag{2}$$

where $\mathbf{u}$ is the Darcy velocity, p is pressure, k is permeability, μ is the viscosity of the fluid, and $\mathbf{f}$ is an L_2 function. Note that this form is equivalent to the pressure form of the saturated flow equation

$$-\nabla\left(\frac{k}{\mu}\nabla p\right) = g,$$

with $g = \nabla \cdot \mathbf{f}$. We formulate the problem in this manner to take advantage of *grad-div* stabilization schemes, which add stability to the discrete form of the governing equations while simultaneously improving mass conservation.[8] Analysis by Olshankii and Reuskin[8] show the addition of the term does not alter the solution to the continuous problem but does improve the stability of the associated bilinear form. They also show that the addition of the term improves the error bounds associated with the discrete solution while simultaneously improving the convergence properties of the iterative solution algorithm.

There have been numerous discussions on what may be considered the best discretization technique for a set of governing equations. Any final assessment of the suitability of a given discretization often depends on the problem at hand. The Taylor-Hood ((P_2, P_1)) approximating pair has been a popular choice for numerical simulation of Stokes and Navier-Stokes flows. Thus, many finite element software packages include Taylor-Hood elements, and the *grad-div* stabilization technique has been easy to implement and analyze for those governing equations.

Given the similarity between the Stokes equations and the form of the governing equations specified in Equations (1) and (2), it seems natural to consider a Taylor-Hood approximating pair for the solution to the saturated flow equations, enhanced with a *grad-div* stabilization scheme to improve discrete mass conservation. The stabilization term, $-\gamma\nabla(\nabla \cdot \mathbf{u})$, $\gamma > 0$, is added to the continuity equation for the Stokes problems, but in our discussion will be added to the Darcy equation 2 above. We note that the addition of this term adds a positive contribution to the variational forms of the governing equations.

Another set of divergence-free elements has also gained more attention recently. The divergence-free Scott-Vogelius elements[3,15,16,21] have been used to compute and analyze discrete solutions of the Stokes and Navier-Stokes equations. These elements use the $\left(\left(P_k, P_{k-1}^{disc}\right)\right)$ approximating pair for velocity and pressure and have been shown to be *inf-sup* stable with optimal approximation properties under mild restrictions.[1,10,20]

Recent analysis[3] has shown that the improvement in mass conservation provided by *grad-div* stabilization scales on the order of the stabilizaton parameter $\gamma > 0$, and in fact the discrete stabilized solution approaches the discrete Scott-Vogelius solution as the stabilization parameter γ tends to ∞. Our goal in this work is to compare three methods for producing velocity fields that adhere to local mass conservation principles. Specifically, we have chosen to study discretizations of Equations (1) and (2) using Raviart-Thomas elements, both RT_0 and RT_1, Scott-Vogelius elements ($\left(P_2, P_1^{disc}\right)$) on a barycenter refinement of an unstructured mesh, and Taylor-Hood elements using *grad-div* stabilization.

The remainder of this paper is organized as follows. We provide the variational form and appropriate function spaces for the discrete solution in Section 2. Numerical results using three different finite element discretizations are given in Section 3, and our findings are summarized in the last section.

2 MODEL PROBLEM

We consider the following variational problem for finding $\mathbf{u}_h$, p_h:

$$\begin{aligned} a\left(\mathbf{u},\mathbf{v}\right) - b\left(p_h,\mathbf{v}\right) &= F\left(\mathbf{v}\right), \quad \forall \mathbf{v} \in X \\ b\left(q,\mathbf{u}\right) &= 0, \qquad\quad \forall q \in Q \end{aligned}$$

where

$$\begin{aligned} a\left(\mathbf{u},\mathbf{v}\right) &= -\mu k^{-1}\int_\Omega \mathbf{u}\cdot\mathbf{v}\,d\Omega + \gamma\int_\Omega \left(\nabla\cdot\mathbf{u}\right)\left(\nabla\cdot\mathbf{v}\right)\,d\Omega \\ b\left(q,\mathbf{v}\right) &= \int_\Omega q\nabla\cdot\mathbf{v}\,d\Omega \\ F\left(\mathbf{v}\right) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega, \\ Q &= L_0^2\left(\Omega\right), \\ X &= H_0^{div}\left(\Omega\right) \\ &= \left\{\mathbf{v} \;:\; \mathbf{v}\in L^2\left(\Omega\right), \nabla\cdot\mathbf{v}\in L^2\left(\Omega\right), \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\right\}. \end{aligned}$$

We assume the permeability tensor k is invertible.

Note that $\gamma = 0$ above gives the non-stabilized variational form. Coercivity of $a\left(\mathbf{v},\mathbf{v}\right)$ in the discrete variational form for $\gamma = 0$ is guaranteed if $\mathbf{v} \in V_h$ where

$$V_h = \left\{\mathbf{v}\in X_h \;:\; b\left(q,\mathbf{v}\right)\forall q\in Q_h\right\}$$

and $\nabla\cdot\mathbf{v} = 0$; that is, $\mathbf{v}$ is in a divergence free subspace. However, when $\gamma > 0$, we have coercivity of $a\left(\mathbf{v},\mathbf{v}\right)$ without the requirement that $\mathbf{v}$ be divergence-free.

Theorem 2.1 *Let u_h^γ and u_h^0 denote the grad-div stabilized (with parameter γ) Taylor-Hood and Scott-Vogelius velocity approximations to the variational formulation considered above. Then on a fixed bary-center refined mesh u_h^γ converges to u_h^0 as $\gamma\to\infty$ with rate γ^{-1}. That is*

$$\|u_h^\gamma - u_h^0\|_{L^2} \leq \frac{C}{\gamma}.$$

Remark 2.1 *The use of grad-div stabilization improves mass conservation. However, in a general setting over stabilizing may destroy numerical solutions. The convergence result shown for incompressible Stokes-type problems gives sufficient conditions for when a large grad-div stabilization parameter may be used.*[7]

3 NUMERICAL RESULTS

We consider two model problems. In both cases, we compare results against a known analytical solution. This allows us to directly compare convergence rates and discuss mass conservation properties of each discrete element choice. In each case, we assume that k and ν are identity elements. All of the results shown below were computed using the `FreeFEM++` environment.[9]

3.1 Problem 1

This problem was presented as part of an analysis of an algorithm for resolving a coupled Stokes-Darcy problem.[5] The domain is the unit square and we use the specified velocity/pressure pair

$$\mathbf{u} = \left[1 - 2x + x^2 + y - y^2,\ -1 + x + 2y - 2xy\right]$$
$$p = (1-x)y(1-y) - x + x^2 - \frac{x^3}{3} + \frac{1}{6}$$

The discrete solution is set equal to the true solution at the boundary.

Table 1 contains data for the discrete velocity solutions found with Scott-Vogelius elements, Raviart-Thomas elements (RT_0 and RT_1), and grad-div stabilized Taylor-Hood elements ($\gamma = 1$ and $\gamma = 1000$). We consider the divergence errors, discrete solution errors, and the time it takes to compute solutions. In an effort to give a fair comparison meshes were chosen for each element to give approximately the same number of degrees of freedom (approximately 98,000). The Taylor-Hood and Scott-Vogelius solutions were computed on a bary-center refined mesh with h=$\frac{1}{64}$ (prior to refinement), and the mesh spacings for the RT_0 and RT_1 elements were h=$\frac{1}{128}$ and h=$\frac{1}{64}$ respectively.

As expected, the discrete solution error for the RT_0 element is large (O(10^{-1})) while the discrete divergence error is small. However, the SV, RT_1, and TH solutions have small discrete errors. We note that the time required to compute the SV solution appears discouraging. This is due to the large SV pressure space, which is not represented in this table. However, the SV element is able to resolve this flow on a coarser mesh requiring only 6274 velocity degrees of freedom and 1.36 seconds, which indicates that the SV element is a competitive element choice for this problem.

Table 2 compares the TH solutions without stabilization and with stabilization ($\gamma = 1$). The divergence errors for the solutions found without stabilization do not diminish as the mesh is refined, and the asymptotic convergence rate is linear. Standard finite element error analysis shows that the optimal convergence rate is (k+1), where k is the degree of polynomials used to approximate the velocity, which in this case is two. Adding grad-div stabilization improves the divergence error and also achieves the optimal convergence rate of three.

Elements	$\|\nabla \cdot \mathbf{u}_h\|$	$\|\mathbf{u} - \mathbf{u}_h\|$	df	t
SV	1.463e-11	4.333e-12	98818	183.117
RT_0	3.621e-10	0.649369	98816	5.77
RT_1	1.463e-11	4.326e-12	82432	6.022
TH+*grad-div* ($\gamma = 1$)	1.598e-5	6.374e-8	98818	13.286
TH+*grad-div* ($\gamma = 1000$)	1.597e-8	2.143e-8	98818	13.439

Table 1: Errors for Problem 1, divergence-free discrete spaces

3.2 Problem 2

Our second test problem is again polygonal, but with nonvanishing boundary values.[2] The domain is again the unit square and the specified solution is

$$\mathbf{u} = \left[x^2 y - \frac{y^3}{3}, \; \frac{x^3}{3} - y^2 x \right]$$
$$p = \frac{x^3 y}{3} - \frac{y^3 x}{3}.$$

We specify $\mathbf{u} \cdot \mathbf{n}$ on the boundary.

We use this numerical experiment to verify solutions computed with SV elements are optimally accurate, and that TH solutions with grad-div stabilization converge to the SV solution on a fixed bary-center refined mesh as $\gamma \to \infty$. Table 3 shows that the SV solutions to the problem are divergent free and optimally accurate, and Table 4 shows that the stabilized TH solutions converge to the SV solution with rate γ^{-1}.

h^{-1}	$\|\nabla \cdot \mathbf{u}_h\|$	$\|\mathbf{u} - \mathbf{u}_h\|$	rate	$\|\nabla \cdot \mathbf{u}_h\|$	$\|\mathbf{u} - \mathbf{u}_h\|$	rate
4	4.7652	0.1150	-	0.0040	2.57e-4	-
8	5.0181	0.0595	0.95	0.0010	3.25e-5	2.98
16	5.1167	0.0301	0.98	2.55e-4	4.07e-6	2.99
32	5.1505	0.0151	0.99	6.38e-5	5.09e-7	2.98
64	5.1611	0.0075	0.99	1.60e-5	6.37e-8	3.00

Table 2: Divergence and L^2 errors for velocity solutions to problem1 computed with Taylor-Hood elements, (left) without grad-div stabilization, and (right) with grad-div stabilization ($\gamma = 1$)

h^{-1}	$\|\nabla \cdot \mathbf{u}_h\|$	$\|\mathbf{u} - \mathbf{u}_h\|$	rate
4	4.11e-12	9.46e-4	-
8	4.11e-12	1.18e-4	2.99
16	4.39e-12	1.47e-5	3.02
32	4.12e-12	1.84e-6	2.99
64	4.14e-12	2.31e-7	2.99

Table 3: Divergence and L^2 errors for velocity solutions to problem 2 computed with SV elements

γ	$\|\mathbf{u}_h^{SV} - \mathbf{u}_h^{TH,\gamma}\|_{L^2}$	L^2 rate
1	1.1445e-6	-
10	1.18612e-7	0.98
100	1.19044e-8	0.99
1000	1.19102e-9	0.99
10000	1.19171e-10	1.00

Table 4: Convergence rates for TH solutions with grad-div stabilization to SV solution. Problem 2

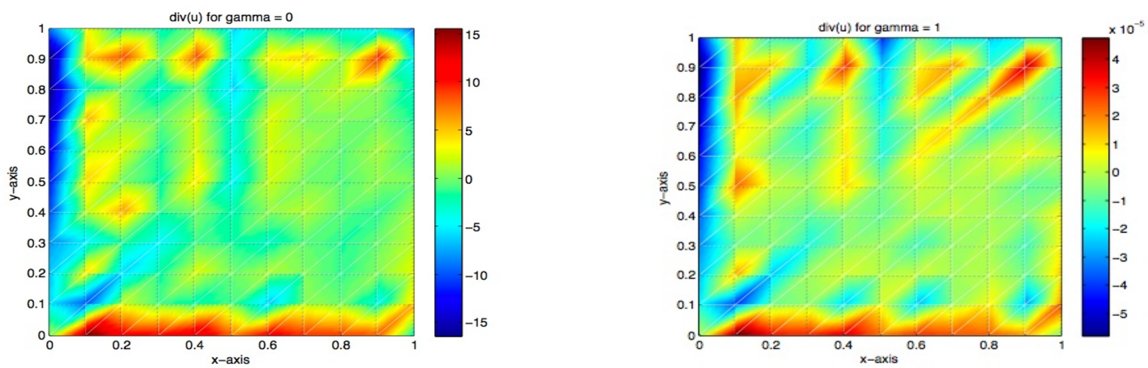

Figure 1: Divergence errors for non-stabilized (left) and stabilized (right) solutions

4 CONCLUSIONS

Clearly, choosing a divergence-free subspace is optimal for computing discrete solutions to flow problems where mass conservation is important. However, given a Taylor-Hood discretization of the velocity/pressure solution, we can improve the mass conservation properties with the simple addition of the *grad-div* stabilization term. Many applications now require the coupling of domains with different physics. There are numerous legacy codes that use a variety of discretization schemes to resolve individual components of a multiphysics problem. A complete rewrite of the software in order to utilize the most recent scheme is often not in order, often due to budgetary and regulatory constraints.

Thus, it is worthwhile to study algorithms that can improve properties of *legacy* discrete solutions with little effort. In this paper, we have provided numerical and theoretical results that suggest the use of the additional *grad-div* stabilization term meets these goals: it is easy to implement in a given code structure, and it significantly reduces the discrete divergence error for the associated porous media velocity field. Scott-Vogelius elements also provide discrete velocity solutions with low discrete divergence errors, with a simple modification of an existing mesh. In both cases, both velocity and pressure are recovered in the solution process.

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