

## SIMULATING GROUNDWATER FLOW IN FRACTURED POROUS ROCK FORMATIONS USING THE ANALYTIC ELEMENT METHOD

Ivan S. P. Marin\*, Edson Wendland\* and Otto D. L. Strack†

\*Universidade de São Paulo

Hydraulics and Sanitation Department, Engineering School of São Carlos  
Av. Trabalhador São-carlense, 400 CP 359 São Carlos, SP CEP 13566-590 Brazil  
e-mail: imarin@sc.usp.br, web page: <http://albatroz.shs.eesc.usp.br/>

† Department of Civil Engineering, University of Minnesota  
500 Pillsbury Drive SE, Minneapolis MN 55455 USA

**Key words:** groundwater simulation, numerical methods, Analytic Element Method, rock fracture

**Summary.** We present the development of an analytic element for simulation of groundwater flow in fractured porous formations, using the Analytic Element Method (AEM), and the development of a matrix method for boundary conditions matching for the AEM. Simulation of groundwater flow in fractured porous media is relevant in different contexts, like the storage of spent nuclear fuel and reactor decommission material in geologic waste repositories. Groundwater flow and transport studies must be conducted to assert the safety of the repositories, as groundwater is the main contaminant vector for biosphere contamination from the stored nuclear material. Inclusion of fractures in the modeling process is paramount, as fractures are usually present in almost all geologic domains, and can propagate groundwater and solutes several orders of magnitude faster than the porous media. Fractures also are characterized as multiscale problems, where different problem scales - from centimeters to kilometers - can influence the behavior of the groundwater flow and the consequent solute transport. The usual groundwater simulation methods, while capable of including fracture phenomena in the simulations, have problems with the scale differences, as they usually depend on domain discretization. The fractures modelled in this work can be filled with porous materials, are considered to be vertical and follow the parallel plates approximation. Exact solutions for one fracture element allowed comparison with the numerical solutions and with fractured generated with the line doublet elements, with satisfactory results. A simulation for a system composed of several fractures with different apertures and lengths is presented.

## 1 INTRODUCTION

Simulation of groundwater flow in fractured porous media is a relevant subject, in the light of recent advances in the geologic repositories context ([5]), for example. Spent nuclear fuel and other decommission residues must be safely stored for long periods of time ([15]), with geologic waste repositories being one of the most viable and studied final destination ([6]). Groundwater flow and transport evaluation must be conducted to assert the safety of the repositories, as groundwater is the main contaminant vector for the stored material. Fractures play a major role in this context, as they are usually present in almost all geologic domains, and can propagate groundwater (and solutes) several orders of magnitude faster than the porous media ([5]). Fractures also are characterized as multiscale problems, with fractures ranging from centimeters to kilometers, can influence groundwater flow and the consequent solute transport. The usual groundwater simulation methods, while capable of including fracture phenomena in the simulations, can present problems with the scale differences because of its dependence on domain discretization. The Analytic Element Method is based on analytic solutions of the groundwater governing equations and does not depend on domain discretization([13, 11]). The Analytic Element Method has been applied in different fields, as wellhead protection area delineation. This work proposes to improve the Analytic Element Method developing an analytic element for flow in fractures, using the recent developments as the use of Cauchy Integrals, combined with the use of the Joukowski Transform ([14]). These new developments allow increased precision on the numerical boundary conditions matching for all elements. The matrix method, developed by ([11]), coupled with the Cauchy Integral formalism was developed because of numerical problems in the boundary condition matching for the fractures using the standard boundary condition matching in the AEM. Exact solutions for one fracture element allowed comparison with the numerical solutions and with fractured generated with the line doublet elements, with satisfactory results.

### 1.1 The Analytic Element Method

The Analytic Element Method, developed by Otto D. L. Strack ([13]), consists in superposing solutions of the governing equations for groundwater flow. Each solution represent an hydrogeologic feature, and the free coefficients on these functions are found, in one of the solution approaches, specifying the boundary conditions for the coordinate points on the elements (the so-called collocation method). In recent years, the method received attention with its use for harmonically heterogeneous aquifers ([8]), among other advances. There are several approaches ([3, 1, 2, 10]) to improve the numeric accuracy or the decrease the computational time required to solve the resulting system of equations. Recently, Strack ([14]) used the Cauchy Integral formula combined with the Joukowski transform to satisfy the boundary conditions for circle inhomogeneities and line doublets.

## 2 Fracture Element

The analytic element for a fracture is based on the work done in ([12]). The fractures are vertical and integrated using the Dupuit Forchheimer approach. Some approximations are made: it's assumed that the shape of the fracture comes from the normal displacement of the sides of a pressurized crack in an infinite elastic medium ([9, 12]). It is also assumed that the fracture aperture is very small compared with its length, so the hydraulic head across the fracture can be considered to be constant, and equal to the hydraulic head of the host rock on the fracture boundaries.

The model for one fracture uses  $(s, n)$  as a local coordinate system with its origin on the center of the fracture. A coordinate transformation to a dimensionless coordinate system  $Z$  is used, having the form

$$z = x + iy \longrightarrow Z = X + iY = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)} \quad (1)$$

where  $z_1$  and  $z_2$  are the initial and final points of the fracture and  $L$  being the length  $|z_2 - z_1|$ . The aperture  $b$  on the local coordinate system  $Z$  is equal to

$$b = b_m \sqrt{1 - X^2} \quad (2)$$

if the fractured is assumed to be filled with a porous material.

The equations of motion are obtained from the parallel plates approximation ([4]). The discharge in the direction of the fracture  $s$ ,  $Q_s$ , is equal to

$$Q_s(s) = -k^* b \frac{\partial \phi}{\partial s} \quad (3)$$

Considering a complex potential  $\Omega(z)$ , formed by the discharge potential  $\Phi(z)$  and the stream function  $\Psi(z)$ , the total discharge,  $W(z)$ , is equal to the derivative of the complex potential  $\Omega(z)$  ([13]):

$$\Omega(z) = \Phi(z) + \Psi(z), W(z) = -\frac{d\Omega(z)}{dz} = Q_x - iQ_y \quad (4)$$

The angle between the coordinate axis  $(s, n)$  and  $(x, y)$  is  $\alpha$ . The derivative of the discharge potential  $\Phi$  in the direction of the fracture,  $s$ , is equivalent to the real part of the total discharge  $W(z)$ , rotated by the angle  $\alpha$ :

$$\Re(W(z)e^{i\alpha}) = \frac{d\Phi}{ds} \quad (5)$$

The discharge on the fracture is equal to the difference on the of the stream function on each side of the fracture, denoted by  $\Psi^+$  and  $\Psi^-$ :

$$Q_s = -(\Psi^+ - \Psi^-) = -k^* b \frac{\partial \phi}{\partial s} = \frac{k^* b}{k} \frac{\partial \Phi}{\partial s} \quad (6)$$

where  $k$  is the hydraulic conductivity of the porous medium. This way, equation (6) can be expressed as

$$\frac{k^*b}{k} \Re \left( W(z)e^{i\alpha} \right) = (\Psi^+ - \Psi^-) \quad (7)$$

### 3 COMPLEX POTENTIAL FOR SEVERAL FRACTURES

The complex potential for more than one fracture can be described using a series expansion form:

$$\Omega(Z)_j = \sum_n a_{j,n} \left( Z + \sqrt{Z-1}\sqrt{Z+1} \right)^n \quad (8)$$

The coefficients  $a_{j,n}$  must be determined so the boundary conditions can be satisfied.

#### 3.1 Determining the coefficients

The Joukowski transform takes a line in the  $Z$  plane to a circle on the  $\chi$  plane. It is defined as

$$Z = \frac{1}{2} \left( \chi + \frac{1}{\chi} \right), \quad \chi = Z + \sqrt{Z-1}\sqrt{Z+1} \quad (9)$$

Using the Joukowski transformation on the jump of the potential for the fracture  $\mu = \Psi^+ - \Psi^-$  and on the fracture aperture  $b$  given by equation (2), the aperture is equal to  $b = b_m \sin(\theta)$ , so the boundary condition (7) can be rewritten as

$$\sum_{n=1}^N a_{jn} \sin(n\theta) = \frac{k^*b^* \sin(\theta)}{k} \Re \left( W(z)e^{i\alpha_j} \right) \quad (10)$$

The equation (10) has the same structure as a Fourier series expansion, and the Fourier orthogonality ([7]) can be used to obtain the unknown coefficients, noting that the interval is restricted to  $[0, \pi]$  for the integral run only once over the element:

$$a_{jn} = \int_0^\pi \frac{k_j^*b_j^* \sin(\theta)}{k} \Re \left( W(z)e^{i\alpha_j} \right) \sin(n\theta_j) d\theta_j \quad (11)$$

The total discharge on the right hand side is the discharge of all the elements present on the system, including the fracture  $j$ .

### 4 MATRIX METHOD

A different method from the common methods in AEM was devised using a matrix form to calculate the coefficients on (11). The so-called matrix method has the general form of a linear system

$$Ax = c \quad (12)$$

$$A_{i,j} = \sum_{j,m}^{M,N} \left( \int_a^b H_{j,m} U_{i,n} dU_{i,n} - \frac{\delta_{i,j} \delta_{m,n}}{P_{i,n}} \right) \quad (13)$$

where, for the fracture,

$$\sin(\theta) \sum_k^M \sum_{n=1}^N \Re \left( \frac{\left( Z_k - \sqrt{Z_k - 1} \sqrt{Z_k + 1} \right)^m dZ_k}{dZ_k} \frac{dZ_k}{dz} \right) \quad (14)$$

$$U_{i,n} = \sin(n\theta_i) \quad (15)$$

$$dU_{i,n} = d\theta_i \quad (16)$$

$$P_{i,n} = \frac{1}{\pi} \left( \frac{K^+ b^+}{K} \right) \quad (17)$$

and for the known vector  $c$

$$c_{i,n} = \int_0^\pi \Re \left( Q_0 e^{-i(\beta_{uf} + \alpha)} \right) \sin(\theta_i) \sin n\theta_i d\theta_i \quad (18)$$

The matrix form is used to solve for different fracture systems. The computational time effort is concentrated on calculating the integrals (11) to formulate the matrix, but this operation is done only once. The time used for solving the linear system is negligible. After constructing the matrix for the system to be solved, any linear system solver can be applied. In case of elements without self-interaction the diagonal elements are equal for each element, and equal to  $P_{i,n}$ , for each element. This matrix form can be generalized to any analytic element, as it has been done for the line doublet and circular analytic element [11].

## 5 RESULTS

The simulations here presented were implemented using Python and run on a desktop computer. The simulated system is composed of 10 fractures with different apertures and hydraulic conductivities, described on table 1. The system includes a regional uniform flow field, and has a hydraulic conductivity of . The element series expansion order is 5 for all elements.

The system depicted on figure 1 is one fracture with the uniform flow with an angle of  $\pi/4$  with respect to the horizontal, and an uniform regional flow of  $Q_0 = 1x10^{-8}m^2/s$ . The background hydraulic conductivity is  $K_b = 1x10^{-9}m/s$ . The figure 2 has the same global parameters.

## 6 CONCLUSIONS

- An Analytic Element for rock fractures has been developed.
- A matrix method to solve for the boundary conditions for the fracture has been developed.
- The matrix method can be applied to other analytic elements.

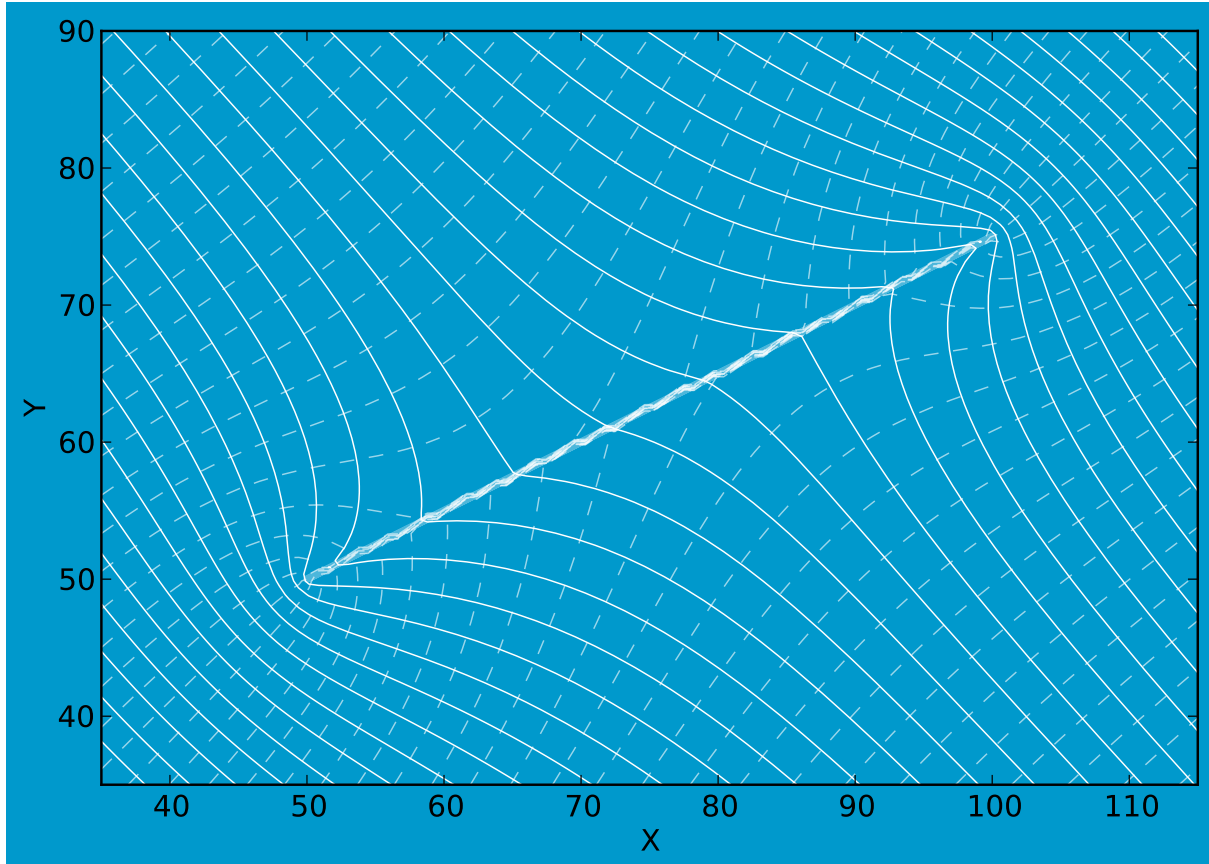


Figure 1: Isolines for head (solid lines) and streamlines (dashed lines) for one fracture in a uniform regional flow.

- Several fractures can be simulated with high efficiency on a desktop computer, using Python.

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	$K^+(M/s)$	$b(m)$	$n$
Fracture 1	0,000074	0,0003	5
Fracture 2	0,000033	0,0002	5
Fracture 3	0,0002	0,0005	5
Fracture 4	0,000033	0,0002	5
Fracture 5	0,000002	0,00005	5
Fracture 5	0,000013	0,0004	5
Fracture 6	0,0002	0,0004	5
Fracture 7	0,00029	0,0006	5
Fracture 8	0,00053	0,0008	5
Fracture 9	0,0002	0,0005	5
Fracture 10	0,0002	0,0005	5

Table 1: Parameter of simulation with fractures with variation on orientation, aperture and hydraulic conductivity, calculated using the cubic law.

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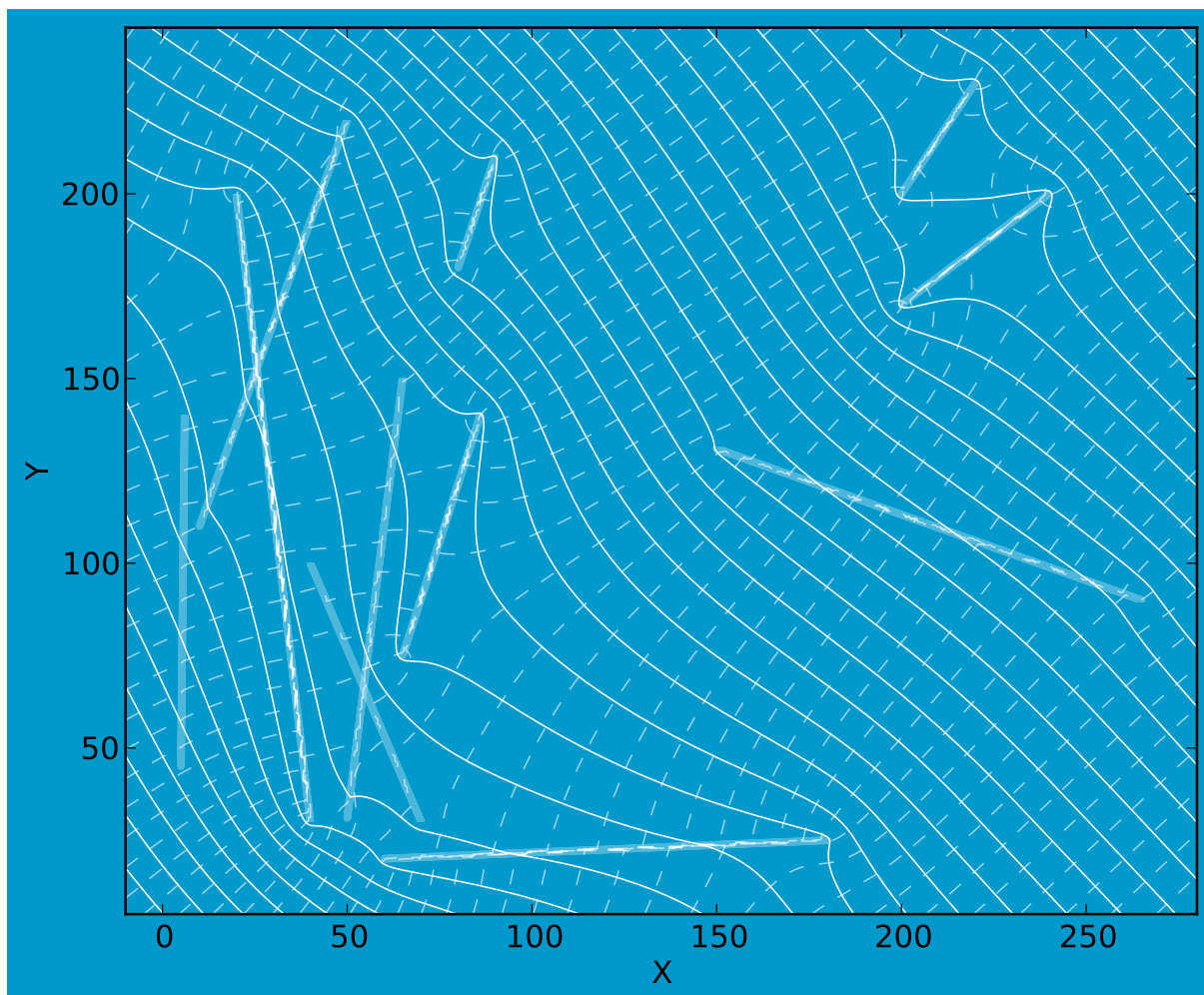


Figure 2: Isolines for head (solid lines) and streamlines (dashed lines) for one fracture in an uniform regional flow.

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